STABLE AVERAGES OF CENTRAL VALUES OF RANKIN-SELBERG L-FUNCTIONS: SOME NEW VARIANTS

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Abstract. As shown by Michel-Ramakrishan (2007) and later generalized by Feigon-Whitehouse (2008), there are "stable" formulas for the average central L-value of the Rankin-Selberg convolutions of some holomorphic forms of fixed even weight and large level against a fixed imaginary quadratic theta series. We obtain exact finite formulas for twisted first moments of Rankin-Selberg L-values in much greater generality and prove analogous "stable" formulas when one considers either arbitrary modular twists of large prime power level or real dihedral twists of odd type associated to a Hecke character of mixed signature.

Contents

The special values $L(f,s)$ of L-functions attached to automorphic forms f are of fundamental arithmetic interest; for instance, such values (often conjecturally) carry information concerning the arithmetic of number fields (the class number formula) and elliptic curves (the Birch and Swinnerton-Dyer conjecture). Motivated by this interest, a basic problem in modern number theory is to study the behavior of such values as f traverses a family of automorphic forms. Some typical problems of interest are to

(1) show that $L(f,s)$ is non-vanishing for at least one (or several) such f,
(2) show that $L(f,s)$ satisfies a nontrivial upper bound in terms of s and the conductor of f (the subconvexity problem), and
(3) study the (possibly twisted) moments of $f \mapsto L(f,s)$; such study has often served as technical input in approaches to the above two problems.

In this paper we consider the family of Rankin-Selberg L-values $L(f \otimes g, s)$ where g is a fixed holomorphic modular form on $\mathrm{GL}(2)/\mathbb{Q}$ and f traverses a family of holomorphic cusp forms of fixed weight, level and nebentypus. We are motivated by work of Michel-Ramakrishnan [10] and later Feigon-Whitehouse [2], who show for certain dihedral forms g arising from idele class characters on imaginary quadratic fields that there

are *finite* formulas for the twisted first moments of central values $f \mapsto L(f \otimes g, \frac{1}{2})$ that simplify considerably, reducing to just one or two terms, when the level of the family to which f belongs is taken to be sufficiently large. One application of such finite formulas that would be inaccessible with an inexact asymptotic formula is to show that there exist f for which the algebraic part of $L(f \otimes g, \frac{1}{2})$ is non-vanishing modulo a prime p. The present work stemmed from a desire to understand better the scope and generality of such phenomena. Because the methods of [10] and [2] make essential use of the restriction that g be dihedral by invoking respectively the Gross-Zagier formula and Waldspurger's formula, we wondered whether the results obtained are likewise exclusive to dihedral g or if they extend to general modular forms g. Our aim in this paper is to show that they do in fact hold quite generally.

Before stating our own results, let us recall in more detail the relevant results of Michel-Ramakrishnan [10]:

(1) Let $-D$ be a negative odd fundamental discriminant, let Ψ be a class group character of $\mathbb{Q}(\sqrt{-D})$, and let g_Ψ be the weight 1 theta series of level D and nebentypus $\chi_{-D} = (D|\cdot)$ attached to Ψ. Let N be a rational prime that is inert in $\mathbb{Q}(\sqrt{-D})$, let k be a positive even integer, and let f traverse an orthogonal basis for the space of newforms of weight k on $\Gamma_0(N)$. Then there is a simple finite formula [10, Thm 1] for the twisted first moment of central L-values

$$\sum_f \frac{L(f \otimes g_\Psi, \frac{1}{2})}{\int_{\Gamma_0(N)\backslash\mathbb{H}} |f|^2 y^k \, \frac{dx\,dy}{y^2}} \lambda_m(f),$$

where $\lambda_m(f) m^{(k-1)/2}$ is the mth Fourier coefficient of f and $L(f \otimes g, s)$ is normalized so that it satisfies a functional equation under $s \mapsto 1-s$. We have spelled out the Petersson norm explicitly here because we shall use a different normalization later in the paper (see (14)).

(2) Moreover, the formula in question becomes astonishingly simple in the so-called "stable range" $N > mD$, in which case all but one or two of its terms vanish; for instance, if $k \geq 4$ and $N > mD$ then one has [10, Cor 1]

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_f \frac{L(f \otimes g_\Psi, \frac{1}{2})}{\int_{\Gamma_0(N)\backslash\mathbb{H}} |f|^2 y^k \, \frac{dx\,dy}{y^2}} \lambda_m(f) = 2\frac{\lambda_m(g)}{m^{1/2}} L(\chi_{-D}, 1),$$

where $\lambda_m(g) m^{(l-1)/2}$ is the mth Fourier coefficient of g and $L(\chi_{-D}, 1) = \sum_{n \geq 1} \chi_{-D}(n)/n = 2\pi h / w D^{1/2}$ with h the class number of $\mathbb{Q}(\sqrt{-D})$ and w the order of its unit group.

As an application, the authors of [10] derive some hybrid subconvexity, non-vanishing and non-vanishing mod p results for N and D in certain ranges; while conceivably the subconvexity and non-vanishing results could have also been derived with a non-exact asymptotic formula having a $o(1)$ term, the non-vanishing mod p results relied crucially on the finiteness of the formula. Note also that while subconvex bounds for $L(f \otimes g, \frac{1}{2})$ are known in generality [11] when either f or g is fixed, the results of [10] address the case that f and g vary simultaneously while satisfying the constraint $(kD)^\delta \ll N \ll D(kD)^{-\delta}$ for some fixed $\delta > 0$. Feigon-Whitehouse [2] generalized many of these results to the context of holomorphic Hilbert modular forms of squarefree level $\mathfrak{N}$ over a totally real number field F averaged against a fixed theta series associated to an idele class character on a CM extension K/F under certain additional conditions such as that $\mathfrak{N}$ be a squarefree product of primes that are inert in K.

As indicated above, our aim in this paper is to prove analogues of (1) and (2) for general (not necessarily dihedral) holomorphic forms g on $\mathrm{GL}(2)/\mathbb{Q}$. To give a flavor for the results we obtain, we begin by stating one of the simplest (and to the author, most surprising) consequences. Denote by $S_k(N, \chi)$ the space of holomorphic cusp forms of weight k, level N and nebentypus χ, and let $S_k(N) = S_k(N, \chi_0)$ where χ_0 is the principal character of modulus N. For $f = \sum a_n n^{(k-1)/2} q^n \in S_k(N, \chi)$ and $g = \sum b_n n^{(l-1)/2} q^n \in S_l(D, \varepsilon)$

($q = \exp(2\pi i z)$), we define

$$L(f \otimes g, s) = L(\chi\varepsilon, 2s) \sum_{n \geq 1} \frac{a_n b_n}{n^s} \tag{1}$$

for $\mathrm{Re}(s) > 1$ and in general by meromorphic continuation [13, 14, 7, 9]; note that while this ad hoc definition has classical precedent as in [9], it may differ by some bad Euler factors from the canonical normalization when f and g are newforms. The critical line for $L(f \otimes g, s)$ is $\mathrm{Re}(s) = 1/2$, and $L(\chi\varepsilon, s) = \sum_{n \geq 1} \chi(n)\varepsilon(n)n^{-s}$ ($\mathrm{Re}(s) > 1$) is the Dirichlet L-function; note that if for instance χ_0 is the principal character mod N, then $L(\chi_0\varepsilon, s)$ is $L(\varepsilon, s)$ without the Euler factors at N.

Theorem 0.1. *Let l be an even positive integer, let k be an odd positive integer such that $k > l$, and let χ be a primitive Dirichlet character of conductor N. Let $\mathcal{F}$ be an orthogonal basis of $S_k(N, \chi)$ with respect to the Petersson inner product. Then for each fixed $g = \sum b_n n^{(l-1)/2} q^n \in S_l(1)$, we have*

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in \mathcal{F}} \frac{L(f \otimes g, \frac{1}{2})}{\int_{\Gamma_0(N)\backslash\mathbb{H}} |f|^2 y^k \, \frac{dx\, dy}{y^2}} = b_1 L(\chi, 1). \tag{2}$$

For example, Theorem 0.1 applies when g is the modular discriminant of weight 12 and f traverses any space of cusp forms of weight 13 and primitive nebentypus χ. It shows immediately that for some f as above, the algebraic part of $L(f \otimes g, \frac{1}{2})$ is non-vanishing modulo any prime of $\overline{\mathbb{Q}}$ for which that of $L(\chi, 1)$ is non-vanishing. If one postulates the nonnegativity of $L(f \otimes g, \frac{1}{2})$ then Theorem 0.1 gives a hybrid subconvex bound for N, χ, k, l as above provided that $(k - l) \gg (kN)^{1+\delta}$ for some $\delta > 0$.

Our main results may be summarized as follows.

- When $g \in S_l(D, \varepsilon)$ is an arbitrary fixed holomorphic cusp form of squarefree level D, we obtain finite formulas along the lines of assertion (1) above for the twisted first moments of $S_k(N, \chi) \ni f \mapsto L(f \otimes g, s)$ whenever $k - l - 2s$ is a nonnegative even integer and $2s < k + l$ (Theorem 1.2). Note that this condition holds for $s = 1/2$ if and only if $k > l$ and the parities of k and l are opposite, in which case $s = 1/2$ is a critical value of $L(f \otimes g, s)$ in the sense of Deligne.
- We show that assertion (2) above extends without essential modification to arbitrary cusp forms g with primitive nebentypus and squarefree level (Theorem 2.4). For instance, while the methods of [10] and [2] apply only to dihedral forms coming from idele class characters on imaginary quadratic fields, we obtain analogous results for the (holomorphic) theta series attached to finite order mixed signature idele class characters on real quadratic fields (Theorem 1.9).
- Assertion (2) above says that the finite formula for the twisted first moment simplifies considerably when the level of the varying form f is sufficiently large compared to that of the fixed form g. We observe a new phenomenon: such simplification occurs also when the level of f is sufficiently divisible by prime divisors of the level of g (Theorem 1.10). Note that in the works [10] and [2], the levels of f and g are always taken to be relatively prime.

Our analysis makes use of the results and method of Goldfeld-Zhang [3], who compute the kernel of the linear map $f \mapsto L(f \otimes g, s)$ in some generality. They suggest in their paper that their results may have some applications for special values of $L(f \otimes g, s)$, and we consider our work in that spirit.

Our paper is organized as follows. In §1.1 we state our general result, which requires a fair amount of notational baggage; the reader is encouraged to skim this section on a first reading and to look instead at §1.2, in which some simple but representative examples are spelled out. In the remainder of §1 we report on some numerical checks of our formulas and describe some of the applications mentioned above. In §2 we give proofs.

1. Results

1.1. **Main result.** In this section we state our main result, from which all others shall follow; we spell out some special cases that require less notational overhead in §1.2. Throughout this paper we let k, l, N, D be positive integers and χ mod N, ε mod D Dirichlet characters such that

$$k \geq 2,\ D \text{ is squarefree},\ \chi(-1) = (-1)^k, \text{ and } \varepsilon(-1) = (-1)^l. \tag{3}$$

The group $\mathrm{GL}_2(\mathbb{R})^+$ acts by fractional linear transformations on the upper half-plane $\mathbb{H} = \{x + iy : y > 0\}$ in the usual way. Recall the weight k slash operator on functions $f : \mathbb{H} \to \mathbb{C}$: for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})^+$, the function $f|_k\alpha$ sends z to $\det(\alpha)^{k/2}(cz+d)^{-k}f(\alpha z)$. The space $S_k(N, \chi)$ of holomorphic cusp forms of weight k, level N and nebentypus χ consists of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ that satisfy

$$f|_k\gamma = \chi(d)f \quad \text{ for all } \gamma = \begin{bmatrix} * & * \\ * & d \end{bmatrix} \in \Gamma_0(N) = \mathrm{SL}_2(\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

and vanish at the cusps of $\Gamma_0(N)$; the space $S_l(D, \varepsilon)$ is defined analogously. For $g \in S_l(D, \varepsilon)$ we write

$$g(z) = \sum_{m=1}^{\infty} b_m m^{(l-1)/2} q^m, \quad q = e^{2\pi i z}, \tag{4}$$

so that the Fourier coefficient b_m of g is normalized so that the Deligne bound reads $|b_p/b_1| \leq 2$ when g is a newform. The cusps of $\Gamma_0(D)$ are indexed by the factorizations of D as a product $D = \delta\delta'$ of positive integers δ and δ'. The scaling matrix for the cusp indexed by δ is

$$W_\delta = \begin{bmatrix} * & * \\ \delta & \delta' \end{bmatrix} \begin{bmatrix} \delta & \\ & 1 \end{bmatrix} \quad \text{with } \begin{bmatrix} * & * \\ \delta & \delta' \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{5}$$

The matrix W_δ is uniquely determined up to left-multiplication by $\Gamma_0(D)$. We write

$$g|W_\delta(z) = \sum_{m=1}^{\infty} b_m^\delta m^{(l-1)/2} q^m \tag{6}$$

for the Fourier coefficients of g at the cusp indexed by δ, so that in particular $b_m^1 = b_m$. This notation for the Fourier coefficients of g at the cusps of $\Gamma_0(D)$ will be in effect throughout the paper. In the special case that ε is primitive and g is a normalized newform, Atkin-Li [1] obtained a formula for the coefficients b_n^D, which we collect here for convenience:

Theorem 1.1. *Suppose that ε is primitive and that $g = \sum_{n \geq 1} b_n n^{\frac{l-1}{2}} q^n \in S_l(D, \varepsilon)$ is a newform with $b_1 = 1$. Then $|b_D| = 1$ and*

$$b_n^D = \varepsilon(-1)\frac{\tau(\varepsilon)}{D^{1/2}}\overline{b_D b_n} \quad \textit{for all } n \geq 1. \tag{7}$$

Proof. One has $W_D \in \Gamma_1(D)\left(\begin{smallmatrix} 0 & -1 \\ D & 0 \end{smallmatrix}\right)$, so the multiplicity-one theorem implies $b_n^D = \overline{\eta(g) b_n}$ for some scalar $\overline{\eta(g)}$; according to [5, Thm 6.29], [6, Prop 14.15] and [5, 6.89], we have $\eta(g) = \tau(\overline{\varepsilon}) b_D D^{-1/2}$. Using that $\overline{\tau(\overline{\varepsilon})} = \varepsilon(-1)\tau(\varepsilon)$, we obtain the claimed formula. Note that Atkin-Li actually consider the operator $h \mapsto h|(-W_D) = (-1)^l h|W_D$, so one must take care in citing their results. $\square$

It will be convenient to define a scaled Petersson inner product on $S_k(N, \chi)$ by the formula

$$(f, g) = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \int_{\Gamma_0(N)\backslash\mathbb{H}} \overline{f(z)} g(z) y^k \, \frac{dx\, dy}{y^2}. \tag{8}$$

By abuse of notation, we write $\sum_{f \in S_k(N,\chi)}$ for the sum over all f in any fixed orthogonal basis of $S_k(N, \chi)$ with respect to the inner product (14).

Recall the definition (1) of $L(f \otimes g, s)$. Our basic object of study is for any k, N, χ as above, $m \in \mathbb{N}$ and $g \in S_l(D, \varepsilon)$ the twisted first moment

$$\mathcal{M}_s(k, N, \chi, m, g) := \sum_{f \in S_k(N,\chi)} \frac{L(f \otimes g, s)}{(f, f)} \overline{\lambda_m(f)}, \tag{9}$$

where $\lambda_m(f) m^{(k-1)/2}$ is the mth Fourier coefficient of f. Note that the definition (9) is independent of the choice of orthogonal basis of $S_k(N, \chi)$. Letting $S_k^{\#}(N, \chi)$ denote the subspace of newforms in $S_k(N, \chi)$ and $\sum_{f \in S_k^{\#}(N,\chi)}$ the sum over f in an orthonormal basis thereof, we similarly define

$$\mathcal{M}_s^{\#}(k, N, \chi, m, g) := \sum_{f \in S_k^{\#}(N,\chi)} \frac{L(f \otimes g, s)}{(f, f)} \overline{\lambda_m(f)}. \tag{10}$$

For $s \in \mathbb{C}$, let $|.|^s : \mathbb{N} \to \mathbb{C}$ denote the multiplicative function $n \mapsto n^s$. For a function $\xi : \mathbb{N} \to \mathbb{C}$ we let $\sigma[\xi]$ denote its convolution with the constant function 1, so that

$$\sigma[\xi](n) = \sum_{d|n} \xi(d). \tag{11}$$

Here and always the sum is taken over the positive divisors d of n. In the special case that ξ is a product of a Dirichlet character with some complex power of $|.|$, it will be convenient to denote by $\sigma[\xi](0)$ the value at $s = 0$ of the meromorphic continuation of the function $s \mapsto \sigma[\xi|.|^{-s}](0)$, for which the series definition applies when $\mathrm{Re}(s) \gg 1$. For example, we have $\sigma[\chi\varepsilon|.|^{-s}](0) = L(\chi\varepsilon, s)$ for any $s \in \mathbb{C}$.

Now suppose that ξ is a Dirichlet character. We let $L_N(\xi, s)$ denote the product of its Euler factors at primes dividing N and $L^N(\xi, s)$ the product of the rest of its Euler factors, so that for $\mathrm{Re}(s) > 1$,

$$L(\xi, s) = \sum_{n \geq 1} \frac{\xi(n)}{n^s} = L_N(\xi, s) L^N(\xi, s), \quad L_N(\xi, s) = \prod_{p|n} (1 - \xi(p) p^{-s})^{-1}.$$

If the modulus of ξ factors as mm' for some relatively prime positive integers m and m', we may write $\xi = \xi_m \xi_{m'}$ where ξ_m has modulus m and $\xi_{m'}$ has modulus m'. For example, our character ε of modulus D factors as $\varepsilon = \varepsilon_\delta \varepsilon_{\delta'}$ for any factorization $D = \delta\delta'$ and also as a product $\prod_{p|D} \varepsilon_p$ over the prime divisors of D. If ξ is primitive of conductor m, let

$$\tau(\xi) = \sum_{a \in (\mathbb{Z}/q)^*} \xi(a) e_m(a)$$

denote its Gauss sum; here and always $e_m : \mathbb{Z}/m \to S^1$ is the additive character

$$e_m(a) = e^{2\pi i a/m}.$$

Recall that our fixed cusp form g has squarefree level D. Let δ be a positive divisor of D and $\delta' = D/\delta$ its complement in D. To each such δ we associate the primitive character ξ of conductor q that induces the character $\chi\varepsilon_{\delta'}$ mod $[N, \delta']$; to keep the notation uncluttered, we suppress the dependence on δ of ξ and q. We adopt the convention that $\xi(0) = 1$ if $q = 1$ and $\xi(0) = 0$ if $q \neq 1$. For each nonzero integer A we define a factorization $A = A_1 A_2$ where $0 < A_1 | q^\infty$ (i.e., A_1 is a positive product of divisors of q) and $(A_2, q) = 1$; for $A = 0$, we take $A_1 = 1$ and $A_2 = 0$. Note that such factorizations depend upon q, and hence upon δ, but

we again suppress this dependence. Our δ-dependent notation may be summarized as

$$\begin{aligned}
&\delta\delta' = D, \quad \varepsilon = \varepsilon_\delta \varepsilon_{\delta'} \text{ with } \varepsilon_\delta \bmod \delta, \varepsilon_{\delta'} \bmod \delta', \\
&\xi \bmod q \text{ primitive } \rightsquigarrow \chi\varepsilon_{\delta'} \bmod [N, \delta'], \\
&\xi(0) = 1 \text{ if } q = 1 \text{ and } 0 \text{ otherwise}, \\
&0 \neq A = A_1 A_2 \text{ with } 0 < A_1 | q^\infty, (A_2, q) = 1, \\
&0 = A = A_1 A_2 \text{ with } A_1 = 1, A_2 = 0.
\end{aligned}$$

Theorem 1.2. *Let $k, N, \chi, l, D, \varepsilon$ satisfy our usual assumptions* (3)*. Choose $s \in \mathbb{C}$ such that $k - l - 2s$ is a nonnegative even integer and $2s < k + l$. Then for any $g \in S_l(D, \varepsilon)$ and $m \in \mathbb{N}$, we have*

$$\mathcal{M}_s(k, N, \chi, m, g) = L(\chi\varepsilon, 2s) \left(\frac{b_m}{m^s} + 2\pi i^{-k} \sum_{\delta | D} T_s^\delta \sum_{\substack{n=1 \\ N_1 | (m\delta - n)_1 q}}^{m\delta} \frac{b_n^\delta}{n^{1-s}} I_s\left(\frac{m\delta}{n}\right) S_s^\delta(m\delta - n) \right), \tag{12}$$

where with $M := [N, \delta']_2$ we have

$$\begin{aligned}
T_s^\delta &= \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \frac{i^l \chi(\delta) \tau(\xi) (\varepsilon_\delta |.|^{-2s})(q)}{\varepsilon_\delta(\delta')} \frac{(\varepsilon_\delta \xi |.|^{1-2s})(M)}{L^M(\varepsilon_\delta \xi, 2s)} \\
S_s^\delta(x) &= (\varepsilon_\delta |.|^{1-2s})(x_1) \overline{\xi}(x_2) \sum_{e | (M, x_2)} \mu\left(\frac{M}{e}\right) \frac{e}{M} \sigma[\varepsilon_\delta \xi |.|^{1-2s}]\left(\frac{x_2}{e}\right), \\
I_s(y) &= \begin{cases} y^{\frac{1-k}{2}} (y-1)^{\frac{k-l}{2} - 1 + s} \dfrac{\Gamma\left(\frac{l+k}{2} - s\right)}{\Gamma(l)\Gamma\left(\frac{k-l}{2} + s\right)} F\left(\begin{matrix} \frac{l-k}{2} + 1 - s, \frac{l-k}{2} + s \\ l \end{matrix}; \dfrac{1}{1-y}\right) & y > 1, \\ 0 & y = 1, s \neq 1/2, \\ i^{l-k+1}/2 & y = 1, s = 1/2 \end{cases}
\end{aligned}$$

with $F = {}_2F_1$ the Gauss hypergeometric function.

Remark 1. Under the hypotheses for $k-l-2s$ assumed in the statement of Theorem 1.2, the hypergeometric function appearing in the definition of $I_s(y)$ for $y > 1$ is a rational function of y with rational coefficients (that may be expressed in terms of associated Legendre functions of the first kind). Precisely, if $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol and we set $r = \frac{k-l}{2} - s \in \mathbb{Z}_{\geq 0}$, then

$$F\left(\begin{matrix} \frac{l-k}{2} + 1 - s, \frac{l-k}{2} + s \\ l \end{matrix}; x\right) = \sum_{n=0}^{r} \frac{(-r+1-2s)_n (-r)_n}{(l)_n (1)_n} x^n. \tag{13}$$

For example, if $k - l = 1$ and $s = 1/2$, then (13) is identically 1. Thus the RHS of (12) is an explicit finite expression whose computation reduces to that of Dirichlet L-values and Gauss sums.

Remark 2. Theorem 1.2 gives a formula for the twisted first moment over a basis of all cusp forms, not just newforms. In certain applications we have $\mathcal{M}_s^\#(k, N, \chi, m, g) = \mathcal{M}_s(k, N, \chi, m, g)$; for example, this happens if there are no oldforms in $S_k(N, \chi)$, or if for any oldform $f \in S_k(N, \chi)$ the functional equation implies $L(f \otimes g, 1/2) = 0$. In that case Theorem 1.2 provides (tautologically) an average over newforms.

Remark 3. When $S_k(N, \chi)$ is one-dimensional and spanned by f, Theorem 1.2 gives an exact formula for some values of $L(f \otimes g, s)$. In general, one can use linear algebra to recover exact finite formulas for $L(f \otimes g, s)/(f, f)$ from those given for the twisted moments $\mathcal{M}_s(k, N, \chi, m, g)$ for a sufficiently large set of integers m.

Remark 4. One can obtain a similar finite formula with an additional explicit term when g is non-cuspidal, but for technical reasons we have not carried this out (see §2.2).

1.2. **Examples.** In this section we spell out some ready-to-use deductions of Theorem 1.2 in special cases for which less notational overhead is required. Recall the definition of I_s from the statement of Theorem 1.2.

Corollary 1.3. *Suppose that $s = 1/2$, k is odd, $N > 1$, χ is primitive, l is even, $D = 1$, and $k > l$. Let $g \in S_l(D) = S_l(1)$. Then*

$$\mathcal{M}_s^{\#}(k, N, \chi, m, g) = L(\chi, 1)\frac{b_m}{m^{1/2}} + \frac{2\pi i^{l-k}\tau(\chi)}{N}\sum_{n=1}^{m-1}\frac{b_n}{n^{1/2}}I_{1/2}\left(\frac{m}{n}\right)\sigma[\chi](m-n). \tag{14}$$

Proof. We apply Theorem 1.2. Under the stated conditions, we have

$$T^1_{1/2} = \frac{i^l\tau(\chi)}{NL(\chi,1)}, \quad S^1_{1/2}(0) = 0, \quad S^1_{1/2}(x) = \sigma[\chi](x) \text{ for } x \neq 0.$$

Note also that $b_n^1 = b_n$. The primitivity of χ implies that there are no oldforms in $S_k(N,\chi)$, so $\mathcal{M}_s^{\#}(\cdots) = \mathcal{M}_s(\cdots)$ and the claim follows. □

Remark 5. When $m = 1$, the sum over n in (14) is empty, so we recover the statement of Theorem 0.1.

Corollary 1.4. *Suppose that $s = 1/2$, k is even, N is prime, $\chi = 1$, l is odd, D is prime, ε is primitive, $k > l$, $N \neq D$ and $g \in S_l(D,\varepsilon)$ is a Hecke eigenform with $b_1 = 1$. Then*

$$\begin{aligned}\mathcal{M}_s(k,N,\chi,m,g) \quad &= \quad \left(1 - \frac{\varepsilon(N)}{N}\right)L(\varepsilon,1)\frac{b_m}{m^{1/2}} + \varepsilon(N)\frac{\tau(\varepsilon)^2\overline{b_D^2}}{D}\left(1-\frac{1}{N}\right)L(\overline{\varepsilon},1)\frac{\overline{b_m}}{m^{1/2}} \\ &+ \quad \frac{2\pi i^{l-k}\tau(\overline{\varepsilon})\varepsilon(N)}{D}(S_1 + S_D),\end{aligned} \tag{15}$$

where

$$S_1 = \sum_{n=1}^{m-1}\frac{b_n}{n^{1/2}}I_{1/2}\left(\frac{m}{n}\right)\left(\overline{\varepsilon}(N)\sigma[\overline{\varepsilon}]\left(\frac{m-n}{N}\right) - \frac{\sigma[\overline{\varepsilon}](m-n)}{N}\right),$$

$$S_D = -D^{1/2}b_D\sum_{n=1}^{mD-1}\frac{\overline{b_n}}{n^{1/2}}I_{1/2}\left(\frac{mD}{n}\right)\left(\sigma[\varepsilon]\left(\frac{m-n}{N}\right) - \frac{\sigma[\varepsilon](m-n)}{N}\right)$$

Proof. The formulas in Theorem 1.2 show that

$$T^1_{1/2} = \frac{i^l\tau(\varepsilon)\varepsilon(N)}{DL^N(\varepsilon,1)}, \quad S^1_{1/2}(0) = 0, \quad S^1_{1/2}(x) = \overline{\varepsilon}(N)\sigma[\overline{\varepsilon}]\left(\frac{x}{N}\right) - \frac{\sigma[\overline{\varepsilon}](x)}{N} \text{ for } x \neq 0,$$

$$T^D_{1/2} = \frac{i^l\varepsilon(N)}{L^N(\varepsilon,1)}, \quad S^D_{1/2}(0) = \left(1 - \frac{1}{N}\right)L(\varepsilon,0), \quad S^D_{1/2}(x) = \sigma[\varepsilon]\left(\frac{x}{N}\right) - \frac{\sigma[\varepsilon](x)}{N} \text{ for } x \neq 0,$$

and $I_{1/2}(1) = i^{l-k+1}/2$; in deducing the above formula for $S^1_{1/2}(x)$ from that given by Theorem 1.2, we have used that $\sigma[\varepsilon](x) = \sigma[\varepsilon](xy)$ if y is a product of factors of D and that $\sigma[\varepsilon](x) = \overline{\varepsilon}(x)\sigma[\overline{\varepsilon}](x)$. Moreover, since g is an eigenform with primitive nebentypus, a formula of Atkin-Li [1] (as quoted in Theorem 1.1) shows that $b_n^D = -\tau(\varepsilon)D^{-1/2}\overline{b_D b_n}$. The terms indexed by n in our formula (12) for which $n \neq mD$ contribute the second line of the claimed formula (10), while $n = mD$ contributes the second term on the RHS of (10) after applying the functional equation $L(\varepsilon,0) = (\pi i)^{-1}\tau(\varepsilon)L(\overline{\varepsilon},1)$ and evaluating $i^{2l-2k} = -1$. □

Remark 6. For applications in which it is known in advance that $L(f \otimes g, 1/2) = 0$ for all forms $f \in S_k(N)$ that come from a form of lower level, the related Theorem 2.4 (in which N is not required to be prime) may be more useful.

Corollary 1.5. *Suppose that $s = 1/2$, k is odd, N is prime, χ is primitive, l is even, D is prime, $\varepsilon = 1$, $k > l$, $N \neq D$, and $g \in S_l(D, \varepsilon)$ is a Hecke eigenform with $b_1 = 1$. For a nonzero integer A write $A = A_1 A_2$ with A_1 a power of the prime N and $(A_2, N) = 1$, and write $C = 2\pi i^{l-k}\tau(\chi)\chi(D)/N$ for brevity. Then*

$$\begin{aligned}\mathcal{M}^{\#}_{1/2}(k, N, \chi, m, g) &= \left(1 - \frac{\chi(D)}{D}\right) L(\chi, 1) \frac{b_m}{m^{1/2}} \\ &+ C \sum_{1 \leq n < m} \frac{b_n}{n^{1/2}} P_{1/2}\left(\frac{m}{n}\right) \overline{\chi}((m-n)_2) \left[\sigma[\chi]\left(\frac{(m-n)_2}{D}\right) - \frac{1}{D}\sigma[\chi]((m-n)_2)\right] \\ &+ C \sum_{1 \leq n < mD} \frac{\mu(D) b_{nD} D^{1/2}}{n^{1/2}} P_{1/2}\left(\frac{mD}{n}\right) \overline{\chi}((mD-n)_2) \sum_{\substack{d \mid (mD-n)_2 \\ (d,D)=1}} \chi(d).\end{aligned}$$

Proof. Follows from Theorem 1.2 in the same manner as the previous two corollaries, using that $b_n^D = \mu(D) b_{nD} D^{1/2}$ when D is squarefree and $\varepsilon = 1$ [6, Proposition 14.16]. Here we have $\mu(D) = -1$ because D is assumed to be prime. $\square$

1.3. **Numerical verification.** In this section we carry out some simple numerical tests of our formulas by choosing triples (k, N, χ) for which $\dim S_k(N, \chi) = 1$, say $S_k(N, \chi) = \langle f \rangle$, so that if $L(f \otimes g, s) \neq 0$, then Theorem 1.2 implies

$$\overline{\lambda_m(f)/\lambda_1(f)} = \frac{\dfrac{b_m}{m^s} + 2\pi i^{-k} \displaystyle\sum_{\delta \mid D} T_s^\delta \sum_{n=1}^{m\delta} \frac{b_n^\delta}{n^{1-s}} I_s\left(\frac{m\delta}{n}\right) S_s^\delta(m\delta - n)}{b_1 + 2\pi i^{-k} \displaystyle\sum_{\delta \mid D} T_s^\delta \sum_{n=1}^{\delta} \frac{b_n^\delta}{n^{1-s}} I_s\left(\frac{\delta}{n}\right) S_s^\delta(\delta - n)} \tag{16}$$

Because the RHS of (16) contains the terms b_m/m^s and b_1 which pop out immediately in the proof (from the diagonal term in the Petersson formula), this seems like a reasonable check of the correctness of Theorem 1.2. The computations below were performed with the computer algebra package SAGE [15].

Example 1.6. Take $(l, D, \varepsilon) = (7, 7, \chi_{-7})$. Then $S_l(D, \varepsilon)$ is three-dimensional with a Hecke basis $\{g_1, g_2, g_3\}$ where $g_1 = q + 9q^2 + 17q^4 - 343q^7 - \cdots$ is self-dual with rational Fourier coefficients and $g_2 = \overline{g_3} = q - 8q^2 + (\alpha + 8)q^3 + (-\alpha - 8)q^5 + (-8\alpha - 64)q^6 + \cdots$ have coefficients in an imaginary quadratic extension of $\mathbb{Q}$, where α satisfies $\alpha^2 + 16\alpha + 2104 = 0$. Take $(k, N, \chi) = (8, 3, 1)$. Then $S_8(3, 1) = S_8(3)$ is one-dimensional, spanned by the newform $f = \sum a(n)q^n = q + 6q^2 - 27q^3 - 92q^4 + 390q^5 - 162q^6 - \cdots$. Let $\nu_i(m)$ $(i = 1, 2, 3)$ denote the right hand side of (15) when $g = g_i$. It is easy to compute $\nu_i(m)$ numerically; we find that $\nu_1(1), \ldots, \nu_1(6)$ are approximately 2.243, 1.189, -1.295, -1.612, 3.130 -0.686 and that $\nu_2(1), \ldots, \nu_2(6)$ are approximately $0.362 + 0.861i$, $0.192 + 0.456i$, $-0.209 - 0.497i$, $-0.260 - 0.619i$, $0.505 + 1.202i$, $-0.110 - 0.263i$, and $\nu_3(m) = \overline{\nu_2(m)}$. Now set $a_i(n) = n^{(k-1)/2}\nu_i(m)/\nu_i(1)$. Since $S_k(N)$ is one-dimensional, we should have $a_i(n) = a(n)$ if our formula is correct. Indeed, we find that $a_i(1), \ldots, a_i(6)$ are $1.000, 6.000, -27.000, -92.000, 390.000, -162.000$ for each i; in fact, $a_i(n)$ and $a(n)$ differ by less than 10^{-13} according to our computation using floating point precision. Of course we have neglected here the important fact that the $\nu_i(m)$ are explicit sums with terms in a cyclotomic extension of $\mathbb{Q}$ (up to a multiple of π). In this case we have $(f, f)^{-1} L(f \otimes g_1, 1/2) = \nu_1(1) = \pi(648/2401)\sqrt{7}$, for example.

Example 1.7. Let $g \in S_2(11)$ be the weight two cusp form corresponding to the elliptic curve of conductor 11, thus $g = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - \cdots$. Take $(k, N, \chi) = (5, 7, \chi_{-7})$, so that $S_5(7, \chi_{-7})$ is one-dimensional and spanned by $f = \sum a(n)q^n = q + q^2 - 15q^4 + 49q^7 - 31q^8 + \cdots$. Let $\nu(m)$ denote the right hand side of the formula given by Corollary 1.4. Our formula then asserts that $(f, f)^{-1} L(f \otimes g, 1/2) = \nu(1)$. Computing numerically, we find that $\nu(1) = \pi(6/121)\sqrt{7}$ and that the numbers $\nu(1), \ldots, \nu(8)$ are

approximately 0.412, 0.103, 0.000, -0.386, -3.330, 1.942, 0.412, -0.199 with normalized ratios $a_1(n) = n^{(k-1)/2}\nu(n)/\nu(1)$ approximately 1.000, 1.000, 0.000, -15.000, 0.000, 0.000, 49.000, -31.000, in fact that $a_1(n) = a(n)$ to within 10^{-13} (but of course we could have computed this exactly as well).

Example 1.8. We give an example where $\chi \neq \overline{\chi}$. Let $g = q - q^2 - q^4 - 2q^5 + 4q^7 + \cdots \in S_2(17)$ be the weight two cusp form corresponding to the elliptic curve of conductor 17, let χ be the primitive character of conductor 11 for which $\chi(2) = e^{2\pi i/10}$, take $(k, N, \chi) = (3, 11, \chi)$, and let f be the normalized newform that spans the one-dimensional space $S_3(11, \chi)$. If $\zeta = e^{2\pi i/10}$, then we have

$$\begin{aligned} f &= q + (-\zeta^3 + 2\zeta^2 - 2\zeta)q^2 + (2\zeta^3 - 3\zeta^2 + 3\zeta - 2)q^3 \\ &\quad + (-4\zeta^2 + 3\zeta - 4)q^4 + 4\zeta^2 q^5 + (\zeta^3 + 3\zeta^2 - 7\zeta + 6)q^6 + \cdots \\ &\approx 1.000 + (-0.690 - 0.224i)q + (-1.118 + 0.812i)q^2 + (-2.809 - 2.040i)q^3 \\ &\quad + (1.236 + 3.804i)q^4 + (0.954 - 0.310i)q^5 + (5.854 - 8.057i)q^6 + \cdots . \end{aligned}$$

As in the previous examples, let $\nu(m)$ denote the right hand side of (1) and $a_1(n) = n^{(k-1)/2}\nu(n)/\nu(1)$. Using that $L(\chi, 1) = \pi\sqrt{1/5 + i/20}$ we find that $a_1(1), \cdots, a_1(6) \approx 1.000, -0.690+0.224i, -1.118-0.812i, -2.809+2.040i, 1.236-3.804i, 0.954+0.310i, 5.854+0.8057i$, so that $a_1(n) \approx \overline{a(n)}$, as expected.

1.4. **Application: stability for twists by mixed signature real quadratic theta series.** Let $K/\mathbb{Q}$ be a real quadratic field, ξ a finite-order Hecke character on K, and g_ξ the corresponding theta series. Popa [12] has studied the Rankin-Selberg L-value $L(f \otimes g_\xi, 1/2)$ when f is a newform of trivial central character and ξ is trivial on $\mathbb{A}_{\mathbb{Q}}^*$, in which case g_ξ is a Maass form. On the other hand, suppose instead that ξ is a finite order character with mixed signature at the infinite places ∞_1, ∞_2 of K, so that as representations of $\mathbb{R}^*$ we have $\{\xi_{\infty_1}, \xi_{\infty_2}\} = \{\mathbf{1}, \mathrm{sgn}\}$. Then g_ξ is a holomorphic cusp form of weight 1.

Remark 7. One can heuristically explain why g_ξ is holomorphic of weight 1 in the following two ways. First, the Galois representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ induced by ξ regarded as a character of $\mathrm{Gal}(K^{ab}/K) \cong \overline{K^* K_{\infty+}^*}\backslash\mathbb{A}_K^*$ is odd (here $K_{\infty+}^*$ is the connected component of the identity in $(K \otimes_{\mathbb{Q}} \mathbb{R})^*$). Second, the gamma factor for the L-function of ξ is $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s)$, where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The gamma factor for a holomorphic cusp form of weight l is $\Gamma_{\mathbb{C}}(\frac{l-1}{2} + s)$, while that for a Maass cusp form of eigenvalue $1/4 + \nu^2$ is $\Gamma_{\mathbb{R}}(s-\nu)\Gamma_{\mathbb{R}}(s+\nu)$. Since the lift g_ξ is characterized by the relation $L(\xi, s) = L(g_\xi, s)$, we see by comparing the gamma factors that it must be holomorphic of weight $l = 1$. See [5, 12.3] for further discussion.

As the following theorem shows, the "stability" result for imaginary quadratic theta series described in the introduction carries over without essential modification to the mixed signature real quadratic case.

Theorem 1.9. *Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of odd fundamental discriminant $D > 0$. Let ξ be a finite order Hecke character of K of modulus $\mathfrak{m}$ such that with respect to a fixed real embedding, we have*

$$\xi((\alpha)) = \frac{\alpha}{|\alpha|}$$

for all $\alpha \in K^*$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$. Define the cuspidal weight 1 theta series g_ξ by

$$g_\xi(z) = \sum_{n \geq 1} b_n q^n = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})},$$

where the sum is taken over all nonzero integral ideals $\mathfrak{a}$ in the ring $\mathcal{O}_K$ of integers in K. Suppose that $N_{K/\mathbb{Q}}(\mathfrak{m})$ is squarefree and prime to D, that the Dirichlet character $n \mapsto \xi((n))$ is quadratic of conductor $N_{K/\mathbb{Q}}(\mathfrak{m})$, and that the Fourier coefficients b_n are real. Let ε be the primitive quadratic Dirichlet character of conductor $D \cdot N_{K/\mathbb{Q}}(\mathfrak{m})$ given by $\varepsilon(n) = \chi_D(n)\xi((n))$ for all positive integers n, where χ_D is the primitive

quadratic Dirichlet character associated to the extension K. Let k be even, N a rational prime for which $\varepsilon(-N) = 1$, and $\chi = 1$. Then if $N > mDN_{K/\mathbb{Q}}(\mathfrak{m})$, we have

$$\mathcal{M}^{\#}_{1/2}(k, N, \chi, m, g) = 2\frac{b_m}{m^{1/2}} L(\varepsilon, 1). \tag{17}$$

Remark 8. The quantity on the right hand side of (17) has an arithmetic interpretation. The Fourier coefficient b_m is the sum of ξ taken over the integral ideals in $\mathcal{O}_K$ of norm m. The character ε corresponds to an imaginary quadratic field K' of discriminant $D' = -D \cdot N_{K/\mathbb{Q}}(\mathfrak{m})$. Letting h' denote the class number of $\mathcal{O}_{K'}$ and w' the cardinality of $\mathcal{O}^*_{K'}$, we have

$$L(\varepsilon, 1) = \frac{2\pi h'}{w'|D'|^{1/2}}.$$

Remark 9. In [10], the analogous stability result for imaginary quadratic theta series was applied to obtain non-vanishing and non-vanishing mod p results for the central Rankin-Selberg L-values $L(f \otimes g, 1/2)$. Since the formula (17) is identical in our case, the applications carry over without modification. For instance, with hypotheses as in Theorem 1.9, we find (taking $m = 1$) that as soon as $N > DN_{K/\mathbb{Q}}(\mathfrak{m})$, there exists $f \in S_k(N, \chi)$ for which $L(f \otimes g_\xi, \frac{1}{2}) \neq 0$; under the same conditions, if $p > k + 1$ and p does not divide h', then there exists $f \in S_k(N, \chi)$ for which the algebraic part of $L(f \otimes g_\xi, \frac{1}{2})$ is not divisible by any place of $\overline{\mathbb{Q}}$ above p. The former result can be sharpened; we refer to [10, Thm 3, Thm 4] for details.

1.5. **Application: stability in the vertical sense.** We give one example of this new variant of stability, saving a more general treatment for §2.4.2.

Theorem 1.10. *Suppose that k is even, $\chi = 1$, l is odd, ε is primitive, $k > l$, and that for some prime divisor p of D we have $p^{\alpha+1}|N$ and $p^\alpha \geq mD$. Then*

$$\mathcal{M}_{1/2}(k, N, \chi, m, g) = \frac{b_m}{m^{1/2}} L^N(\varepsilon, 1). \tag{18}$$

Remark 10. In Theorem 1.2 and Theorem 1.10, we have averaged over an orthogonal basis of all cusp forms, including oldforms. We have been able to average over newforms in Theorem 1.9 by imposing the assumption that N is a prime for which $\varepsilon(-N) = 1$: this assumption guarantees that for any oldform f_0, the sign in the functional equation for $L(f_0 \otimes g, s)$ is -1, whence $L(f_0 \otimes g, \frac{1}{2}) = 0$.

In some concrete cases, it is possible to apply Theorem 1.10 directly to compute an average over a basis of newforms. For example, let ε be the primitive character of conductor 3. There is a unique cusp form f (in fact, a newform) of weight 4, level 9, nebentypus ε and a unique form g of weight 1, level 3, nebentypus ε. Since $\mathbb{Q}(\sqrt{-3})$ has 6 units and 1 ideal class, we have $L(\varepsilon, 1) = 2\pi/6\sqrt{3}$, so that taking $m = 1$ in Theorem 1.10 gives

$$\frac{L(f \otimes g, \frac{1}{2})}{(f, f)} = \frac{2\pi}{6\sqrt{3}} \tag{19}$$

(recall our non-standard definition of the Petersson norm (2)).

2. Proofs

2.1. **The basic idea.** Let V be the finite-dimensional inner product space $V = S_k(N, \chi)$ with respect to the (scaled) Petersson inner product (14), which we have normalized to be linear in the second variable. For any linear functional $\ell : V \to \mathbb{C}$, there exists a unique vector $\ell^* \in S_k(N, \chi)$ (called the kernel of ℓ) such that $\ell(f) = (f, \ell^*) = \overline{(\ell^*, f)}$ for all $f \in V$. Let $\{f\}$ be an orthonormal basis of V. Then expanding the kernels $\ell_i^* = \sum \overline{\ell_i(f)} f$, we find $\ell_2(\ell_1^*) = \overline{\ell_1(\ell_2^*)} = \sum \overline{\ell_1(f)} \ell_2(f)$. In particular, defining the expectation $\mathbb{E}[\overline{\ell_1}\ell_2]$ of the product $\overline{\ell_1}\ell_2$ by the formula

$$\mathbb{E}[\overline{\ell_1}\ell_2] = \sum \overline{\ell_1(f)} \ell_2(f), \tag{20}$$

we obtain:

Lemma 2.1. *The definition* (20) *is independent of the choice of orthonormal basis* $\{f\}$*, and satisfies*

$$\mathbb{E}[\overline{\ell_1}\ell_2] = \overline{\ell_1(\ell_2^*)} = \ell_2(\ell_1^*).$$

For example, let $\lambda_m \in V^*$ be the normalized Fourier coefficient

$$\lambda_m : V \ni \sum_{n\geq 1} a_n n^{(k-1)/2} q^n \mapsto a_m \in \mathbb{C}. \tag{21}$$

Then the kernel λ_m^* is a multiple of the mth holomorphic Poincare series, and (for $k \geq 2$) the Petersson formula expresses $\mathbb{E}[\overline{\lambda_m}\lambda_n] = \lambda_n(\lambda_m^*) = \overline{\lambda_m(\lambda_n^*)}$ as $\delta_{mn} + \Delta_{mn}$ where δ_{mn} is the Kronecker delta and Δ_{mn} is a sum of Kloosterman sums weighted by Bessel functions (see (35)).

Now fix a modular form $g = \sum b_n n^{(l-1)/2} q^n \in S_l(D, \varepsilon)$ and define $L_s \in V^*$ for $\mathrm{Re}(s) > 1$ by the series

$$L_s(f) = \sum \frac{\lambda_n(f) b_n}{n^s} \tag{22}$$

and for general $s \in \mathbb{C}$ by meromorphic continuation; then

$$\mathcal{M}_s(k, N, \chi, m, g) = L(\chi\varepsilon, 2s)\mathbb{E}[\overline{\lambda_m} L_s]. \tag{23}$$

Our plan in this paper is to study the moments $\mathbb{E}[\overline{\lambda_m} L_s]$ via the Petersson formula: for $\mathrm{Re}(s) > 1$, we have

$$\mathbb{E}[\overline{\lambda_m} L_s] = \sum_{n\geq 1} \frac{b_n \mathbb{E}[\overline{\lambda_m}\lambda_n]}{n^s} = \frac{b_m}{m^s} + \sum_{n\geq 1} \frac{b_n \Delta_{mn}}{n^s}. \tag{24}$$

Theorem 1.2 is obtained by writing

$$\Delta_{mn} = \int_{a/c\in\mathbb{Q}, w\in\mathbb{C}} e_c(na) n^{-w} \, d\mu_m(a/c, w)$$

for some measure μ_m, applying Voronoi summation to $\sum_n b_n e_c(na) n^{-s-w}$, and observing that all but finitely many of the terms in the resulting expression for $\mathbb{E}[\overline{\lambda_m} L_s]$ are zero.

To put this method in context, we apply Lemma 2.1 to λ_m and L_s; doing so gives a relationship between the mth twisted first moment of $f \mapsto L(f \otimes g, s)$, the mth Fourier coefficient of the kernel of the linear map $f \mapsto L(f \otimes g, s)$, and the L-value $L(\lambda_m^* \otimes g, s)$:

Lemma 2.2. $\mathbb{E}[\overline{\lambda_m} L_s] = \overline{\lambda_m(L_s^*)} = L_s(\lambda_m^*)$.

When g is the theta series attached to a class group character of an imaginary quadratic field, Gross-Zagier [4, Ch. IV] compute $\lambda_m(L_{1/2}^*)$ roughly as follows: the Rankin-Selberg method shows that $L(f \otimes g, s) = (f, gE_s)_{\Gamma_0(ND)\backslash\mathbb{H}}$ for a non-holomorphic Eisenstein series E_s of level ND, so the kernel $L_{1/2}^*$ is the holomorphic projection of $\mathrm{Tr}_N^{ND}(gE_{1/2})$.

Goldfeld-Zhang [3] use the identity $\overline{\lambda_m(L_s^*)} = L_s(\lambda_m^*)$ of Lemma 2.2 and the Petersson formula for $\lambda_n(\lambda_m^*)$ to compute $L_s(\lambda_m^*) = \sum_n b_n n^{-s} \lambda_n(\lambda_m^*)$ for any $s \in \mathbb{C}$, with the motivation of giving a simpler and more general derivation of Gross-Zagier's result. The basic idea of writing the Fourier coefficient of such a kernel function as an infinite linear combination of Poincaré series had been raised (but not carried out) by Zagier [16] in the context of $L(\mathrm{sym}^2 f, s)$ (see especially [16, p. 38]).

When g is the theta series attached to a class group character of an imaginary quadratic field, Michel-Ramakrishnan used the identity $\mathbb{E}[\overline{\lambda_m} L_{1/2}] = \overline{\lambda_m(L_{1/2}^*)}$ of Lemma 2.2 and the Gross-Zagier computation of $\lambda_m(L_{1/2}^*)$ to study the twisted first moments $\mathbb{E}[\overline{\lambda_m} L_{1/2}]$.

Feigon-Whitehouse [2] generalized some of the work of [10] to the case that f lives in a family of holomorphic Hilbert modular forms over a totally real number field and g is induced by an idele class character of a

CM extension. By Waldspurger's formula, they relate $L(f\otimes g,\frac{1}{2})$ to a toral period of the Jacquet-Langlands correspondent of f on a suitable quaternion algebra, which they then average by a relative trace formula.

Thus the method of Goldfeld-Zhang applies in the situation we consider, but for two reasons their results are not directly applicable here. First, the only essential assumptions we place on k,N,χ and l,D,ε are that D be squarefree; Goldfeld-Zhang impose the most restrictive assumptions that χ be trivial, D squarefree and ε primitive. Second, and more importantly, the formulas that they obtain and some of the calculations in their arguments are not sufficiently detailed for our purposes; we elaborate on this point in the next section.

2.2. **On some results of Goldfeld-Zhang.** The formula for $\mathcal{M}_s(k,N,\chi,m,g)$ asserted by Theorem 1.2 is a special case of the following more general formula for $\mathbb{E}[\overline{\lambda_m}L_s]$ without any assumptions on s.

Theorem 2.3. *Let k,N,χ,l,D,ε satisfy* (3)*. Then for any $g\in S_l(D,\varepsilon)$ and $m\in\mathbb{N}$, we have*

$$\mathbb{E}[\overline{\lambda_m}L_s]=\frac{b_m}{m^s}+2\pi i^{-k}\sum_{\delta|D}T_s^\delta\sum_{\substack{n=1\\ N_1|(m\delta-n)_1q}}^{\infty}\frac{b_n^\delta}{n^{1-s}}I_s\left(\frac{m\delta}{n}\right)S_s^\delta(m\delta-n),\tag{25}$$

as meromorphic functions of s, where T_s^δ and $S_s^\delta(x)$ are as in Theorem 1.2 and for $\mathrm{Re}(s)>1/2$*,*

$$I_s(y)=\int_{\varepsilon-\frac{k-1}{2}-i\infty}^{\varepsilon-\frac{k-1}{2}+i\infty}\frac{G_k(w)}{G_l(s+w)}y^{-w}\,\frac{dw}{2\pi i},\quad G_k(w):=\frac{\Gamma(\frac{k-1}{2}+w)}{\Gamma(\frac{k-1}{2}+1-w)}.\tag{26}$$

We shall prove Theorem 2.3 in §2.3. As in [3, Proposition 8.3], one obtains for $\mathrm{Re}(s)<\frac{k+l}{2}$ that as meromorphic functions of s,

$$I_s(y)=\begin{cases} y^{\frac{k-1}{2}}(1-y)^{\frac{l-k}{2}-1+s}\dfrac{\Gamma(\frac{l+k}{2}-s)}{\Gamma(k)\Gamma(\frac{l-k}{2}+s)}F\begin{pmatrix}\frac{k-l}{2}+s,\frac{k-l}{2}+1-s & ; \dfrac{y}{y-1}\\ k & \end{pmatrix} & y\in(0,1)\\ y^{\frac{1-k}{2}}(y-1)^{\frac{k-l}{2}-1+s}\dfrac{\Gamma(\frac{l+k}{2}-s)}{\Gamma(l)\Gamma(\frac{k-l}{2}+s)}F\begin{pmatrix}\frac{l-k}{2}+1-s,\frac{l-k}{2}+s & ; \dfrac{1}{1-y}\\ l & \end{pmatrix} & y>1\\ \dfrac{\Gamma(\frac{l+k}{2}-s)\Gamma(2s-1)}{\Gamma(\frac{k-l}{2}+s)\Gamma(\frac{l-k}{2}+s)\Gamma(\frac{l+k}{2}-1+s)} & y=1,\end{cases}\tag{27}$$

A direct computation shows that for $y=1$, we have $I_s(1)=0$ unless $s=1/2$, in which case $I_{1/2}(1)=i^{l-k+1}/2$. One now deduces Theorem 1.2 from Theorem 2.3 by noting that $I_s(y)=0$ for $0<y<1$ and s satisfying the conditions of Theorem 2.3: indeed, if $r=\frac{k-l}{2}-s$ is a nonnegative integer and $2s<k+l$, then for $y\in(0,1)$ and $2\mathrm{Re}(z)<k+l$, the function $z\mapsto I_z(y)$ is the product of an entire function with $z\mapsto\Gamma(\frac{l-k}{2}+z)^{-1}$, the latter of which vanishes at $z=s$ because of the condition we have imposed on $k-l-2s$.

Remark 11. There is some subtlety in interpreting Theorem 2.3 when $s=1$. Precisely, note that $S_s^\delta(0)$ unless ξ is the trivial character, in which case $S_s^\delta(0)=L(\varepsilon_\delta,2s-1)\phi(M)/M$, which has a pole at $s=1$ if ε_δ is a principal character. However, if $k-l-2s\in 2\mathbb{Z}_{\geq 0}$, then such a pole will always be cancelled by a zero of $I_s(1)$ at $s=1$, so the resulting expression for $\mathbb{E}[\overline{\lambda_m}L_s]$ still makes sense.

In the special case that $\chi=1$ and ε is primitive, Theorem 2.3 is similar to what one would obtain by combining the results [3, Theorem 6.5], [3, Proposition 7.1], and [3, Proposition 8.3] of Goldfeld-Zhang and applying the identity $L_s(\lambda_m^*)=\mathbb{E}[\overline{\lambda_m}L_s]$ given in Lemma 2.2. The proof we shall give follows their method, but the result that we obtain differs in the following ways.

(1) In (25), we have restricted the sum over $n\geq 1$ by the condition $N_1|(m\delta-n)_1q$, whereas in [3], the stronger condition $N|(m\delta-n)_1q$ is imposed (in [3] one takes $\chi=1$ and ε primitive, so that $q=\delta'$); it is not hard to see from the arguments of [3] that this disagreement is accounted for by a typo.

(2) In [3], i^k appears where we have written i^{-k}. These are equal when k is even, which is the case in [3]; i^{-k} is the correct expression when k is odd.

(3) The factor i^{-l} appears in [3] where we have written i^l; this difference is more serious (giving the wrong answer in the important case that l is odd) and results from the propagation of a sign error, as we shall explain in §2.3.

(4) In the proof of [3, Proposition 8.3] the transformation identity for the hypergeometric function is misapplied, resulting in a formula for $I_s(y)$, $y > 1$ with "k" passed as the third argument to the hypergeometric function rather than "l" as we have written it above.

(5) When g is non-cuspidal, a certain non-entire function "$L_g(s, a/c)$" whose poles are determined in [3, Proposition 4.2] is claimed to be entire in the middle of [3, p738]; some formal manipulations that follow are not valid and ultimately yield a formula that is missing a term under certain circumstances (including the important case that $k = 2$, $l = 1$, and g is the theta series attached to the trivial class group character of an imaginary quadratic extension). We limit ourselves to the case that g is cuspidal so as not to have to worry about this.

(6) At the beginning of [3, §7], our calculations indicate that one should take "$\varepsilon^\delta = \varepsilon$" rather than "$\varepsilon^\delta = \varepsilon_\delta^{-1}\varepsilon_{\delta'}$" for most of the formulas that follow to be correct; thus, most instances of $\varepsilon_{\delta'}$ in our formula for S_s^δ appear as $\varepsilon_{\delta'}^{-1}$ in [3, §7]. (The definition of $S^\delta(s, B)$ in [3, §6] should have "$\varepsilon_{\delta'}^{-1}(\bar{r})$," where $\bar{r}$ denotes the inverse of r modulo c (a multiple of δ'), rather than "$\varepsilon_{\delta'}^{-1}(r)$" as is written; this follows directly from the fact that "$L_g(s + w, \bar{r}/c)$" appears in [3, 6.2] and that "$L_g(s, a/c) = \varepsilon_{\delta'}^{-1}(a) \times \cdots$" by [3, Prop 4.2]. To confuse matters further, there is a typo in the statement of [3, Lemma 5.3]: $\varepsilon_{\delta'}$ should be replaced by $\varepsilon_{\delta'}^{-1}$ in the definition of the Gauss sum. However, it seems that the correct form of that lemma is what is actually used in the rest of the paper.) To avoid confusion, we carry out the relevant calculations in §2.3.

Suppose now that $\chi = 1$ and ε is primitive. Goldfeld-Zhang simplify their analogue of (25) for $L_s(\lambda_m^*)$ under progressively restrictive assumptions: first that $\varepsilon^2 = 1$, and then that $(N, D) = 1$ and g is an imaginary quadratic theta series. We shall make analogous simplifications, but cannot directly use their results: some of the calculations in their arguments are not sufficiently detailed for our purposes, and the expression they ultimately derive for $L_s(\lambda_m^*)$ when g is an imaginary quadratic theta series is inconsistent with the functional equation satisfied by $L(f \otimes g, s)$ for $f \in S_k(N) = S_k(N, 1)$. For completeness, we prove the following variant of [3, Proposition 9.1]. Recall that λ_m^* is the kernel of the linear functional λ_m on the inner product space $S_k(N)$. Let us now write $\lambda_{m,N}^*$ to indicate its dependence on the level N. We say that a form $f \in S_k(N)$ is *of strictly lower level* if there exists a proper divisor M of N such that f factors through the inclusion $S_k(M) \hookrightarrow S_k(N)$.

Theorem 2.4. *Suppose that $\chi = 1$, ε is primitive, and $(N, D) = 1$. Then there exists a linear combination $v \in S_k(N)$ of forms of strictly lower level such that*

$$\begin{aligned} L^N(\varepsilon, 2s) L_s\left(\lambda_{m,N}^* + v\right) &= \frac{b_m}{m^s} L(\varepsilon, 2s) + 2\pi i^{l-k} (\varepsilon|.|^{1-2s})(N) \sum_{\delta | D} \left(\frac{\delta}{4\pi^2}\right)^{1/2-s} \frac{\tau(\varepsilon_{\delta'})}{(\delta')^{2s}} \\ &\cdot \sum_{\substack{n \geq 1 \\ N | (m\delta - n)}} \frac{b_n^\delta}{n^{1-s}} I_s\left(\frac{m\delta}{n}\right) (\varepsilon_\delta|.|^{1-2s})[(m\delta - n)_1] \overline{\varepsilon_{\delta'}}[(m\delta - n)_2] \\ &\cdot \sigma[\varepsilon|.|^{1-2s}]\left(\frac{(m\delta - n)_2}{N}\right). \end{aligned} \tag{28}$$

Proof. If M is a divisor of N, then we shall regard $\lambda^*_{m,M} \in S_k(M)$ as an oldform of lower level in $S_k(N)$. Since $(N, D) = 1$ and $\chi = 1$, we have $N_2 = N$ and $N_1 = 1$ for all divisors δ of D, so we may write (25) as

$$L_s(\lambda^*_{m,N}) = \frac{b_m}{m^s} + \sum_{\delta|D} B_\delta(N) \sum_{e|N} \mu\left(\frac{N}{e}\right) F(\delta, e) \tag{29}$$

where (using that ε is primitive)

$$\begin{aligned}
A(\delta) &= 2\pi i^{l-k} \left(\frac{\delta}{4\pi^2}\right)^{1/2-s} \frac{\tau(\varepsilon_{\delta'})}{(\delta')^{2s}}, \\
B(N) &= \frac{(\varepsilon|.|^{-2s})(N)}{L^N(\varepsilon, 2s)}, \\
C(\delta, n) &= \frac{b^\delta_n}{n^{1-s}} I_s\left(\frac{m\delta}{n}\right) (\varepsilon_\delta|.|^{1-2s})[(m\delta - n)_1] \overline{\varepsilon_{\delta'}}[(m\delta - n)_2], \\
E(\delta, n, e) &= e\, \sigma[\varepsilon|.|^{1-2s}]\left(\frac{(m\delta - n)_2}{e}\right) \quad (:= 0 \text{ unless } e|(m\delta - n)_2), \\
F(\delta, e) &= \sum_n A(\delta) C(\delta, n) E(\delta, n, e).
\end{aligned} \tag{30}$$

We have suppressed the dependence of these expressions on the variables s and m, which we now regard as fixed. Rearranging the sums in (29), we find

$$L_s(\lambda^*_{m,N}) = \frac{b_m}{m^s} + B(N) \sum_{e|N} \mu\left(\frac{N}{e}\right) \sum_{\delta|D} F(\delta, e).$$

Applying inclusion-exclusion, we obtain

$$\begin{aligned}
L_s\left(\sum_{M|N} \frac{B(N)}{B(M)} \lambda^*_{m,N}\right) &= \frac{b_m}{m^s} \sum_{M|N} \frac{B(N)}{B(M)} + B(N) \sum_{\delta|D} \sum_{M|N} \sum_{e|M} \mu\left(\frac{M}{e}\right) F(\delta, e) \\
&= \frac{b_m}{m^s} \sum_{M|N} \frac{B(N)}{B(M)} + B(N) \sum_{\delta|D} F(\delta, N).
\end{aligned} \tag{31}$$

Let us temporarily set $\psi = \varepsilon|.|^{-2s}$. Then

$$\sum_{M|N} \frac{B(N)}{B(M)} = \psi(N) L_N(\varepsilon, 2s) \sum_{M|N} \frac{\prod_{p|M}(1 - \psi(p))}{\psi(M)}.$$

Applying inclusion-exclusion once again, the inner sum is

$$\begin{aligned}
\sum_{M|N} \frac{\prod_{p|M}(1 - \psi(p))}{\psi(M)} &= \sum_{M|N} \psi^{-1}(M) \sum_{e|M} \mu(e)\psi(e) = \sum_{e|N} \psi(e)\mu(e) \sum_{\substack{M|N \\ M \equiv 0(e)}} \psi^{-1}(M) \\
&= \sum_{e|N} \mu(e) \sum_{M|\frac{N}{e}} \psi^{-1}(M) = \sum_{M|N} \psi^{-1}(M) \sum_{e|\frac{N}{M}} \mu(e) \\
&= \psi^{-1}(N),
\end{aligned}$$

so that

$$\sum_{M|N} \frac{B(N)}{B(M)} = L_N(\varepsilon, 2s). \tag{32}$$

Substituting (32) and (30) in (31) gives (28). □

2.3. **Proof of Theorem 2.3.** In this section we generalize and refine the results of Goldfeld-Zhang [3], as explained in §2.2, following their method. Let $g = \sum b_n n^{(l-1)/2} q^n \in S_l(D, \varepsilon)$. The functional equation for the additive twists of $L(g,s)$ is proved (up to a sign; see below) in [3, Proposition 4.2], and asserts the following. Let c be a positive integer and $a \in (\mathbb{Z}/c\mathbb{Z})^*$. Associate to c the decomposition $D = \delta\delta'$ with $\delta' = (c, D) > 0$, and to a the complex number

$$\eta_{a/c} = \frac{i^l}{\varepsilon_\delta(\delta')\varepsilon_{\delta'}(\delta)} \varepsilon_\delta(c)\varepsilon_{\delta'}(a), \tag{33}$$

where $\bar{a} = a^{-1} \in (\mathbb{Z}/c\mathbb{Z})^*$ as usual. Then we have an equality of entire functions

$$\sum_n \frac{b_n e_c(\bar{a}n)}{n^s} = \eta_{a/c} \left(\frac{\delta c^2}{4\pi^2}\right)^{\frac{1}{2}-s} \frac{\Gamma(\frac{l-1}{2}+1-s)}{\Gamma(\frac{l-1}{2}+s)} \sum_n \frac{b_n^\delta e_c(-a\bar{\delta}n)}{n^{1-s}}. \tag{34}$$

In [3, Proposition 4.2] the formula (34) is obtained but with the factor i^l in $\eta_{a/c}$ replaced by i^{-l}. This results from a sign error in [3, page 735, displayed equation 6], where one should write $(-cz)^l$ instead of $(cz)^l$. This propagates to a sign error in the statement of [3, Proposition, 4.2] and the theorems that follow. As a sanity check, take $c = 1$, $a = 0$, $\delta = D, \delta' = 1$. In that case it is a classical theorem of Hecke (see for instance [6, Theorem 14.7]) that the functional equation relating g (with coefficients b_n) to its Fricke involute $g|W_D$ (with coefficients b_n^D) is

$$\sum_n \frac{b_n}{n^s} = i^l D^{1/2-s} \frac{\Gamma(\frac{l-1}{2}+1-s)}{\Gamma(\frac{l-1}{2}+s)} \sum_n \frac{b_n^D}{n^{1-s}},$$

which agrees with (34) as we have written it. The statement and proof of [3, Proposition, 4.2] are otherwise correct.

Write $G_k(w) = \Gamma(\frac{k-1}{2}+w)/\Gamma(\frac{k-1}{2}+1-w)$. The term Δ_{mn} in the Petersson formula $\mathbb{E}[\overline{\lambda_m}\lambda_n] = \delta_{mn} + \Delta_{mn}$ is given by (see e.g. [6, Proposition 14.5])

$$\Delta_{mn} = 2\pi i^{-k} \sum_{\substack{c \geq 1 \\ N|c}} \frac{S_\chi(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{35}$$

where for any $\varepsilon \in (0, 1/2)$

$$S_\chi(m,n;c) = \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \chi(a) e_c(ma + n\bar{a}), \quad J_{k-1}(x) = \int_{(\varepsilon - \frac{k-1}{2})} G_k(w) \left(\frac{x}{2}\right)^{-2w} dw$$

are the Kloosterman sum and Bessel function. For $\mathrm{Re}(s)$ large enough we have

$$\mathbb{E}[\overline{\lambda_m} L_s] = \sum_{n \geq 1} \frac{b_n \mathbb{E}[\overline{\lambda_m}\lambda_n]}{n^s} = \frac{b_m}{m^s} + \sum_{n \geq 1} \frac{b_n \Delta_{mn}}{n^s}, \tag{36}$$

so it remains only to compute $\sum b_n \Delta_{mn} n^{-s}$. Substituting the definition (35) of Δ_{mn} and applying the functional equation (34), we obtain (using here that our additively-twisted L-functions are entire)

$$\begin{aligned}
\sum_n \frac{b_n \Delta_{mn}}{n^s} &= 2\pi i^{-k} \sum_{\substack{c \geq 1 \\ N|c}} \sum_n \frac{S_\chi(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{b_n}{n^s} \\
&= 2\pi i^{-k} \sum_{\substack{c \geq 1 \\ N|c}} \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \frac{\chi(a) e_c(ma)}{2\pi i c} \int_{(\varepsilon - \frac{k-1}{2})} G_k(w) \left(\frac{4\pi^2 m}{c^2}\right)^{-w} \sum_n \frac{b_n e_c(\bar{a}n)}{n^{s+w}}\, dw \\
&= 2\pi i^{-k} \sum_{\delta | D} \left(\frac{\delta}{4\pi^2}\right)^{\frac{1}{2}-s} \sum_n \frac{b_n^\delta}{n^{1-s}} I_s\left(\frac{m\delta}{n}\right) R_s^\delta(m\delta - n), \qquad (37)
\end{aligned}$$

where $I_s(y)$ is as in the statement of Theorem 2.3 and

$$R_s^\delta(x) = \sum_{\substack{c \geq 1 \\ N|c \\ (c,D)=\delta'}} c^{-2s} \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \chi(\delta a) \eta_{\delta a/c} e_c(xa).$$

Substituting the definition (33) of $\eta_{a/c}$, we find

$$R_s^\delta(x) = \frac{i^l \chi(\delta)}{\varepsilon_\delta(\delta')} \sum_{\substack{N|c \geq 1 \\ (c,D)=\delta'}} c^{-2s} \varepsilon_\delta(c) \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \chi\varepsilon_{\delta'}(a) e_c(xa). \qquad (38)$$

Recall from the beginning of §1.1 that we let ξ denote the primitive character of conductor f that induces $\chi\varepsilon_{\delta'}$, that by convention $\xi(0) = 1$ if $f = 1$ and $\xi(0) = 0$ otherwise, that for $A \neq 0$ we write $A = A_1 A_2$ with $0 < A_1 | f^\infty$ and $(A_2, f) = 1$, that for $A = 0$ we take $A_1 = 1$ and $A_2 = 0$, and that we set $M = [N_2, \delta'_2] = [N, \delta']_2$.

Since f divides $[N, \delta']$ which in turn divides c, we have $\chi\varepsilon_{\delta'}(a) = \xi(a)$ whenever $(a, c) = 1$, so that we may write ξ instead of $\chi\varepsilon_{\delta'}$ in (38).

Lemma 2.5. *If $x_1 f = c_1$, then we have*

$$\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \xi(a) e_c(xa) = x_1 \overline{\xi}(x_2) \xi(c_2) \tau(\xi) \sum_{d | (c_2, x_2)} d\mu\left(\frac{c_2}{d}\right), \qquad (39)$$

otherwise the sum on the left hand side of (39) vanishes.

Proof. Suppose first that $x \neq 0$. Let us write $p^\alpha || n$ to denote that a prime p divides n to order exactly α, in which case we also write $\alpha = v_p(n)$. The Chinese remainder theorem shows that $1 \equiv \sum_{p^\alpha || c} \overline{cp^{-a}} cp^{-a}$ (mod c), where $\overline{cp^{-a}}$ is any inverse mod p^α. Thus

$$e_c(xa) = \prod_{p^\alpha || c} e_{p^\alpha}(\overline{cp^{-\alpha}} xa),$$

and we may factor the left hand side of (39) as

$$\left(\prod_{p^\alpha || c_1} \sum_{a \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*} \xi_p(a) e_{p^\alpha}(\overline{cp^{-\alpha}} xa)\right) \sum_{a \in (\mathbb{Z}/c_2\mathbb{Z})^*} e_{c_2}(xa).$$

Similarly,

$$\tau(\xi) = \prod_{p^\beta || f} \sum_{a \in (\mathbb{Z}/p^\beta\mathbb{Z})^*} \xi_p(a) e_{p^\beta}(\overline{fp^{-\beta}} a).$$

The evaluation of the Ramanujan sum

$$\sum_{a \in (\mathbb{Z}/c_2\mathbb{Z})^*} e_{c_2}(xa) = \sum_{d | (c_2, x_2)} d\mu\left(\frac{c_2}{d}\right)$$

is well-known: by Mobius inversion, this amounts to the assertion that $\sum_{a \in \mathbb{Z}/n\mathbb{Z}} e_n(xa)$ is n if $n|x$, 0 otherwise, and that $(c_2, x) = (c_2, x_2)$.

Fix a prime divisor p of c_1 (equivalently of f, since f and c_1 have the same support), and write $\alpha = v_p(c_1)$, $\beta = v_p(f)$. Since ξ_p is primitive of conductor $p^\beta \neq 1$ and the additive character $a \mapsto e_{p^\alpha}(\overline{cp^{-\alpha}} xa)$ has period $p^{\alpha - v_p(x)}$, we see that $\sum_{a \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*} \xi_p(a) e_{p^\alpha}(\overline{cp^{-\alpha}} xa)$ vanishes unless $v_p(x) = \alpha - \beta$, in which case

$$\begin{aligned}
(40) \qquad \sum_{a \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*} \xi_p(a) e_{p^\alpha}(\overline{cp^{-\alpha}} xa) &= p^{\alpha-\beta} \sum_{a \in (\mathbb{Z}/p^\beta\mathbb{Z})^*} \xi_p(a) e_{p^\beta}\left(\overline{cp^{-\alpha}} \frac{x}{p^{\alpha-\beta}} a\right) \\
&= \xi_p\left(\frac{c}{p^\beta x}\right) p^{\alpha-\beta} \sum_{a \in (\mathbb{Z}/p^\beta\mathbb{Z})^*} \xi_p(\overline{fp^{-\beta}} a) e_{p^\beta}(\overline{fp^{-\beta}} a) \\
&= \xi_p\left(\frac{c}{fx}\right) p^{\alpha-\beta} \sum_{a \in (\mathbb{Z}/p^\beta\mathbb{Z})^*} \xi_p(a) e_{p^\beta}(\overline{fp^{-\beta}} a).
\end{aligned}$$

In the second step we made the substitution $(\mathbb{Z}/p^\beta\mathbb{Z})^* \ni a \mapsto \overline{cxf} a$.

Suppose that (40) is nonzero for each $p^\alpha || c_1$. Then $v_p(x) = v_p(c_1) - v_p(f)$ for each $p|c_1$, so that $x_1 f = c_1$, and we obtain

$$\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \xi(a) e_c(xa) = x_1 \xi\left(\frac{c_2}{x_2}\right) \tau(\xi) \sum_{d | (c_2, x_2)} d\mu\left(\frac{c_2}{d}\right),$$

as desired. This completes the proof of Lemma 2.5 when $x \neq 0$. In the remaining case that $x = 0$, the sum (39) vanishes unless $f = 1$, in which case it equals $\sum_{d|c_2} d\mu\left(\frac{c_2}{d}\right)$. Given our conventions, this is what the lemma asserts. □

By (39), we have $R_s^\delta(x) = 0$ unless there exists a positive integer c such that $N_1 | c_1$ and $c_1 = x_1 f$, so let us assume henceforth that $N_1 | x_1 f$. Then

$$\begin{aligned}
(41) \qquad R_s^\delta(x) &= \frac{i^l \chi(\delta)}{\varepsilon_\delta(\delta') \varepsilon_{\delta'}^2(\delta)} \sum_{\substack{N | c \geq 1 \\ (c, D) = \delta'}} (\varepsilon_\delta |.|^{-2s})(c) \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \xi(a) e_c(xa) \\
&= \frac{i^l \chi(\delta)}{\varepsilon_\delta(\delta') \varepsilon_{\delta'}^2(\delta)} \sum_{\substack{N_2 | c_2 \geq 1 \\ (c_2, D) = \delta'_2}} (\varepsilon_\delta |.|^{-2s})(x_1 f c_2) x_1 \overline{\xi}(x_2) \xi(c_2) \tau(\xi) \sum_{d | (c_2, x_2)} d\mu\left(\frac{c_2}{d}\right) \\
&= \frac{i^l \chi(\delta) \tau(\xi) (\varepsilon_\delta |.|^{-2s})(f) (\varepsilon_\delta |.|^{1-2s})(x_1) \overline{\xi}(x_2)}{\varepsilon_\delta(\delta') \varepsilon_{\delta'}^2(\delta)} \sum_{M | c_2 \geq 1} (\varepsilon_\delta \xi |.|^{-2s})(c_2) \sum_{d | (c_2, x_2)} d\mu\left(\frac{c_2}{d}\right).
\end{aligned}$$

We evaluate the sums over c_2, d in the following lemma, where we set $\psi = \varepsilon_\delta \xi |.|^{-2s}$.

Lemma 2.6. *Provided that the following sums converge, we have*

$$\sum_{M|c_2>0} \psi(c_2) \sum_{d|(c_2,x_2)} d\mu\left(\frac{c_2}{d}\right) = \frac{(\psi|.|^1)(M)}{L^M(\psi)} \sum_{e|(M,x_2)} \mu\left(\frac{M}{e}\right) \frac{e}{M}\sigma[\psi|.|^1]\left(\frac{x_2}{e}\right). \tag{42}$$

Proof. Interchanging summation, we have

$$S := \sum_{M|c_2} \psi(c_2) \sum_{d|(c_2,x_2)} d\mu\left(\frac{c_2}{d}\right) = \sum_{d|x_2} d \sum_{[M,d]|c_2} \psi(c_2)\mu\left(\frac{c_2}{d}\right),$$

where $[a,b]$ is the least common multiple of a and b. The maps

$$c_2 \mapsto \left(\frac{Md}{(c_2,Md)}, \frac{c_2}{(c_2,Md)}\right), \quad (e,l) \mapsto \frac{Md}{e}l,$$

give a bijection of sets of natural numbers

$$\{c_2 : [M,d]|c_2\} \leftrightarrow \{(e,l) : e|(M,d), (l,e) = 1, \}$$

so that

$$\begin{aligned} S &= \sum_{d|x_2} d \sum_{e|(M,d)} \sum_{(l,e)=1} \psi\left(\frac{Md}{e}l\right)\mu\left(\frac{M}{e}l\right) = \sum_{d|x_2} d \sum_{e|(M,d)} \psi\left(\frac{Md}{e}\right)\mu\left(\frac{M}{e}\right) \sum_{\substack{(l,e)=1 \\ (l,\frac{M}{e})=1}} \psi(l)\mu(l) \\ &= \frac{1}{L^M(\psi)} \sum_{e|M} \sum_{d|\frac{x_2}{e}} de\, \psi(M)\psi(d)\mu\left(\frac{M}{e}\right) = \frac{(\psi|.|^1)(M)}{L^M(\psi)} \sum_{e|M} \mu\left(\frac{M}{e}\right) \frac{e}{M}\sigma[\psi|.|^1]\left(\frac{x_2}{e}\right), \end{aligned}$$

as desired. $\square$

Taking $\psi = \varepsilon_\delta \xi |.|^{-2s}$ and substituting (42), (41), (37) into (36) gives (25).

Remark 12. There are other choices of coefficients (b_n) that lead to interesting linear functionals $L_s = \sum b_n n^{-s}\lambda_n$. For instance, one could consider the Fourier coefficients of a Maass cusp form. As another example, take $\chi = 1$ and $b_n = \tau(n)$, so that $\zeta^N(2s)L_s(f) = L(f,s)^2$ for $f \in S_k(N)$. Let $E(z,s)$ denote the usual real-analytic Eisenstein series for $SL(2,\mathbb{Z})$, so that

$$\frac{\partial}{\partial s}E(z,s)|_{s=1/2} = y^{1/2}\log y + 4y^{1/2}\sum_{n\geq 1}\tau(n)K_0(2\pi ny)\cos(2\pi nx)$$

and

$$\frac{-1}{2\pi}\frac{\partial^2}{\partial x \partial s}E(z,s)|_{s=1/2} = 4y^{1/2}\sum_{n\geq 1} n\tau(n)K_0(2\pi ny)\sin(2\pi nx).$$

Using these equations, one could derive functional equations for the additive twists

$$\sum_{n\geq 1}\tau(n)e^{2\pi i\alpha n}n^{-s}, \quad \alpha \in \mathbb{Q}$$

and apply the methods of this paper to study the twisted *second* moments of the classical modular L-function $L(f,s)$ (as in [8]).

However, tracing through the proof of Theorem 1.2 given above, one sees that the resulting formula for $\mathbb{E}[\overline{\lambda_m}L_s]$ does not simplify to a finite sum when $s = 1/2$, but rather to an infinite series half of whose terms vanish. In general, when the coefficients b_n are those of some $GL_2/\mathbb{Q}$ automorphic form, the finiteness of such a formula following this approach boils down to an assertion about the locations of the zeros and poles of the Gamma factor in the functional equation for the additive twists of $\sum b_n n^{-s}$. For this reason we have restricted our attention to holomorphic forms g.

2.4. **Proofs of applications.** Recall the notation (9), (10) for the twisted first moments $\mathcal{M}_s(k,N,\chi,m,g)$, $\mathcal{M}_s^{\#}(k,N,\chi,m,g)$. Recall also the "$\delta$-dependent" notation $\delta', \eta \bmod q$, M, $A = A_1A_2$ introduced in §1.1 and used in the statement of Theorem 1.2.

2.4.1. *Real dihedral twists.*

Theorem 2.7. *Suppose that k is even, $\chi = 1$, l is odd, ε is primitive quadratic, $k > l$, and $(N, D) = 1$. Suppose moreover that g is a Hecke eigenform with $b_1 = 1$ and that N, k have been chosen so that for any form $f \in S_k(M) \subset S_k(N)$ of strictly lower level $M|N$, $M \neq N$, we have $L(f \otimes g, \frac{1}{2}) = 0$. For brevity write $x = m\delta - n$. Then*

$$\begin{aligned}(43)\ \mathcal{M}^{\#}_{1/2}(k,N,\chi,m,g) \quad &= \quad \frac{b_m}{m^{1/2}}L(\varepsilon,1) + \varepsilon(-N)\overline{b_D^2}\frac{\overline{b_m}}{m^{1/2}}L(\varepsilon,1) \\ &+ \quad 2\pi i^{l-k}\varepsilon(N)\sum_{\delta|D}\frac{\tau(\varepsilon_{\delta'})}{\delta'}\sum_{\substack{1\le n<m\delta \\ N|(m\delta-n)}}\frac{b_n^{\delta}}{n^{1/2}}P_{1/2}\left(\frac{m\delta}{n}\right)\varepsilon_\delta(x_1)\varepsilon_{\delta'}(x_2)\sigma[\varepsilon]\left(\frac{x_2}{N}\right).\end{aligned}$$

Here $b_n^D = -i\overline{b_{nD}}$.

Proof. Under the given conditions, we have $L_{1/2}(v) = 0$ for all oldforms $v \in V$, so that Theorem 2.4 implies that $\mathcal{M}^{\#}_{1/2}(k,N,\chi,m,g)$ is equal to the sum of

$$\frac{b_m}{m^{1/2}}L(\varepsilon,1) + 2\pi i^{-k}\varepsilon(N)\frac{i^l b^D_{mD}}{(mD)^{1/2}}I_{1/2}(1)L(\varepsilon,0)$$

and the second line of (43). We have $I_{1/2}(1) = i^{l-k+1}/2$ and $b^D_{mD} = -\tau(\varepsilon)\overline{b_{mD^2}}D^{-1/2}$ (see Theorem 1.1). The functional equation $\pi i L(\varepsilon, 0) = \tau(\varepsilon)L(\overline{\varepsilon}, 1)$ shows that

$$2\pi i^{-k}\varepsilon(N)\frac{i^l b^D_{mD}}{(mD)^{1/2}}I_{1/2}(1)L(\varepsilon,0) = \frac{\tau(\varepsilon)^2}{D}\varepsilon(N)\frac{\overline{b_{mD^2}}}{m^{1/2}}L(\overline{\varepsilon},1).$$

We have $\tau(\varepsilon)^2 = D\varepsilon(-1)$ and $\overline{\varepsilon} = \varepsilon$ since ε is primitive quadratic, so the formula (43) follows. □

Corollary 2.8. *With conditions and assumptions as in Theorem 2.7, suppose furthermore that $N > mD$. Then*

$$\mathcal{M}^{\#}_{1/2}(k,N,\chi,m,g) = \frac{b_m}{m^{1/2}}L(\varepsilon,1) + \varepsilon(-N)\overline{b_D^2}\frac{\overline{b_m}}{m^{1/2}}L(\varepsilon,1). \tag{44}$$

Proof. The sum over n in (43) is empty when $N > mD$. □

Proof of Theorem 1.9. Let $g = g_\xi \in S_1(D', \varepsilon)$, $D' = D \cdot N_{K/\mathbb{Q}}(\mathfrak{m})$ be as in the statement of Theorem 1.9. Then g is a normalized Hecke eigenform because ξ is a character and a newform because ξ (or ε) is primitive. Since ε is primitive and g is a normalized newform, Theorem 1.1 implies that $|b_D| = 1$. Since ε is quadratic, we have $\varepsilon = \overline{\varepsilon}$. Since $b_D \in \mathbb{R}$, we have $\overline{b_D^2} = 1$. Since by assumption $\varepsilon(-N) = 1$, we see that Theorem 1.9 follows from (44) provided that we can justify the hypothesis that $L(f \otimes g, 1/2) = 0$ for all forms $f \in S_k(N)$ coming from a lower level. Since N is prime, the only possibility is that $f \in S_k(1)$. We may assume by linearity that f is a normalized Hecke eigenform, so that in particular $\overline{f} = f$, $\overline{g} = g$. Then [9, Theorem 2.2, Example 2] shows that

$$L(f\otimes g, s) = \varepsilon(f\otimes g)(1^2D^2)^{1/2-s}\frac{\gamma(1-s)}{\gamma(s)}L(f\otimes g, 1-s), \tag{45}$$

where

$$\gamma(s) = \Gamma_{\mathbb{C}}\left(s + \frac{|k-l|}{2}\right)\Gamma_{\mathbb{C}}\left(s + \frac{k+l}{2} - 1\right), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s),$$

and

$$\varepsilon(f \otimes g) = \begin{cases} \varepsilon(-1)\chi(D)\varepsilon(1)\eta(f)^2\eta(g)^2 & k \leq l \\ \chi(-1)\chi(D)\varepsilon(1)\eta(f)^2\eta(g)^2 & k > l, \end{cases}$$

with $\eta(f), \eta(g)$ defined by $f|W_1 = (-1)^k \eta(f)\overline{f}$ and $g|W_D = (-1)^l \eta(g)\overline{g}$; here W_1, W_D are the Fricke involutions as defined in §1.1. Since ε is primitive quadratic and $b_D \in \{\pm 1\}$, Theorem 1.1 implies that $\eta(g)^2 = \varepsilon(-1)$. Since f has trivial level, we have $\eta(f) = 1$. Since $k > l$ and $\chi = 1$, we see that $\varepsilon(f \otimes g) = \varepsilon(-1) = -1$. Evaluating the functional equation (45) at the point $s = 1/2$ gives $L(f \otimes g, 1/2) = 0$, as desired. □

Remark 13. Note that the argument just given applies also to cuspidal imaginary quadratic theta series (in which context our calculation of b_{mD}^D remains valid), so we have recovered the cuspidal case of the original stability result of [10].

2.4.2. *Vertical stability.*

Theorem 2.9. *Suppose that $k > l$, $k - l \equiv 1 \pmod 2$, and that N is chosen so that for each divisor $\delta | D$ with $(\delta, N) = 1$ we have $N_1/(N_1, q) \geq \max(m\delta, 2)$. Then $\mathcal{M}_{1/2}(k, N, \chi, m, g) = L(\chi\varepsilon, 1) b_m m^{-1/2}$.*

Proof. Fix a divisor δ of D. If $(\delta, N) > 1$, then $\chi(\delta) = 0$, so $T_{1/2}^{\delta} = 0$. If $N_1/(N_1, q) \geq m\delta$, then the sum over n in (12) is empty with the possible exception of the term indexed by $n = m\delta$, which vanishes because $N_1/(N_1, f) \geq \max(m\delta, 2)$ implies $f > 1$ implies $\eta(0) = 0$, so that $S_s^{\delta}(m\delta - m\delta) = 0$. Thus the claim follows from Theorem 1.2. □

Theorem 2.10. *Suppose that k is even, $\chi = 1$, l is odd, ε is primitive, and $k > l$. Let $(N, D^{\infty}) = \lim_{\alpha \to \infty}(N, D^{\alpha})$, and suppose that*

$$\frac{(N, D^{\infty})}{(N, D)} \geq \max(mD, 2).$$

Then $\mathcal{M}_{1/2}(k, N, \chi, m, g) = L^N(\varepsilon, 1) b_m m^{-1/2}$.

Proof. Let δ be a divisor of D for which $(\delta, N) = 1$. Then $f = \delta'$ and $(N, D) = (N, \delta') = (N, f) = (N_1, f)$, $N_1 = (N, D^{\infty})$, so that $N_1/(N_1, f) = (N, D^{\infty})/(N, D) \geq \max(mD, 2) \geq \max(m\delta, 2)$; the claim then follows from the criterion of Theorem 2.9. □

Proof of Theorem 1.10. The conditions of Theorem 2.10 are satisfied when there exists a prime divisor p of D and $\alpha \geq 1$ for which $p^{\alpha+1} | N$ and $p^{\alpha} \geq mD$. □

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